TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 4, Pages 1371–1386
S 0002-9947(00)02734-3
Article electronically published on December 18, 2000

# BLOCH CONSTANTS IN SEVERAL VARIABLES

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ABSTRACT. We give lower estimates for Bloch's constant for quasiregular holomorphic mappings. A holomorphic mapping of the unit ball $B^n$ into $\mathbf{C}^n$ is $K$-quasiregular if it maps infinitesimal spheres to infinitesimal ellipsoids whose major axes are less than or equal to $K$ times their minor axes. We show that if $f$ is a $K$-quasiregular holomorphic mapping with the normalization $\det f'(0) = 1$, then the image $f(B^n)$ contains a schlicht ball of radius at least $1/12K^{1-1/n}$. This result is best possible in terms of powers of $K$. Also, we extend to several variables an analogous result of Landau for bounded holomorphic functions in the unit disk.

## 1. INTRODUCTION

Since the confirmation of the Bieberbach conjecture by de Branges, it has been opined that finding the precise value of the Bloch constant is the number one problem in the geometric function theory of one complex variable. At the time of this writing, the best estimate seems to be in [2]. For general holomorphic mappings of more than one complex variable, there is no (positive) Bloch constant ([7], [16]); in order to obtain higher dimensional analogs of Bloch's theorem, it is necessary to restrict the class of mappings considered. "The most important and useful special case of the general Bloch theorem is, without a doubt, that of quasiconformal holomorphic mappings in several complex variables" [16, p. 195]. In the present paper, we estimate Bloch's constant for bounded holomorphic mappings and for quasiregular holomorphic mappings. In fact, for more than one complex variable, quasiregular holomorphic mappings are locally biholomorphic [10]. Earlier investigations for holomorphic quasiregular mappings and related classes were done by Bochner [1], Hahn [6], Harris [7], Sakaguchi [13], Takahashi [14], and Wu [16]. If we drop the holomorphicity, there is no Bloch theorem for quasiregular mappings of the ball, but Eremenko [3] has recently obtained a Bloch theorem for entire quasiregular mappings.

For different classes, Bloch constants were studied by Fitzgerald and Gong [4], Graham and Varolin [5], and Liu [8].

We denote the complex plane by $\mathbf{C}$. Let $D = \{z \in \mathbf{C} : |z| < 1\}$ be the unit disk and $D_r$ be the disk in $\mathbf{C}$ with center at the origin and radius $r$. Let

$$\mathbf{C}^n = \{z = (z_1, z_2, \cdots, z_n) : z_1, z_2, \cdots, z_n \in \mathbf{C}\}$$

Received by the editors August 10, 1998.
2000 *Mathematics Subject Classification.* Primary 32H99; Secondary 30C65.
*Key words and phrases.* Bloch constant.
Research supported in part by NSFC(China), NSERC(Canada) and FCAR(Québec).

be the complex space of dimension $n$. We identify a point $z = (z_1, z_2, \cdots, z_n) \in \mathbf{C}^n$ and a column vector formed by complex numbers $z_1, z_2, \cdots, z_n$, and define its length by

$$|z| = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{1/2}.$$

Denote a ball in $\mathbf{C}^n$ with center at $z'$ and radius $r$ by

$$B^n(z', r) = \{z \in \mathbf{C}^n : |z - z'| < r\},$$

and denote the unit ball in $\mathbf{C}^n$ by

$$B^n = \{z \in \mathbf{C}^n : |z| < 1\}.$$

For a mapping $f = (f_1, f_2, \cdots, f_n)$ of a domain in $\mathbf{C}^n$ into $\mathbf{C}^n$, we denote by $\partial f/\partial z_k$ the column vector formed by $\partial f_1/\partial z_k, \partial f_2/\partial z_k, \cdots, \partial f_n/\partial z_k$, and we denote by

$$f' = \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \cdots, \frac{\partial f}{\partial z_n}\right)$$

the matrix formed by these column vectors.

For an $n$ by $n$ matrix $A$, we have the matrix norm

$$\|A\| = \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}$$

and the operator norm

$$|A| = \sup_{x \neq 0} \frac{|Ax|}{|x|} = \max\{|A\theta| : \theta \in \partial B^n\},$$

for which $|A| \leq \|A\| \leq \sqrt{n}|A|$.

The classical theorem of Bloch for holomorphic functions in the disk fails to extend to general holomorphic mappings in the ball of $\mathbf{C}^n$. However, in 1946, Bochner [1] proved that Bloch's theorem does hold for a special class of holomorphic mappings.

For a holomorphic mapping $f$ from the unit ball $B^n$ of $\mathbf{C}^n$ into $\mathbf{C}^n$, $B^n(a, r)$ is called a schlicht ball contained in $f(B^n)$ if there is a domain $G \subset B^n$ such that $f$ maps $G$ biholomorphically onto $B^n(a, r)$. We denote by $\beta_f$ the least upper bound of the radii of all schlicht balls contained in $f(B^n)$ and call this the Bloch radius of $f$. The mapping $f$ is said to be normalized if $\det f'(0) = 1$. For a constant $K \geq 1$, let us say that a holomorphic mapping $f$ from the unit ball $B^n$ of $\mathbf{C}^n$ into $\mathbf{C}^n$ is a Bochner $K$-mapping in $B^n$ if $f$ satisfies the differential inequality

$$\|f'\| \leq K |\det f'|^{1/n}$$

at each $z \in B^n$. Bochner showed that, for each $K \geq 1$ and $n \geq 2$, there is a constant $\beta > 0$ such that, for each normalized Bochner $K$-mapping $f$ in the ball, $\beta_f \geq \beta$. However, Bochner gave no estimate for this Bloch constant.

In 1951, Takahashi [14] estimated the Bloch constant for the class of normalized mappings $f$ satisfying the weaker condition

$$\max_{|z| \leq r} \|f'(z)\| \leq K \max_{|z| \leq r} |\det f'(z)|^{1/n}, \text{ for each } 0 \leq r < 1.$$

Let us call such mappings Takahashi $K$-mappings. For such normalized Takahashi $K$-mappings, Sakaguchi [13], in 1956, improved Takahashi's estimate to

$$\beta \geq \frac{(n-1)^{n-2}}{8K^{2n-1}}.$$

Let $\lambda^2$ and $\Lambda^2$ (with $0 \le \lambda \le \Lambda$) denote the smallest and largest characteristic values of the Hermitian matrix $A^*A$, where $A = f'$ and $A^*$ is the conjugate transpose of $A$. Let us say that the holomorphic mapping $f$ is a Hahn $K$-mapping if

$$\max_{|z|=r} \Lambda(z) \le K \max_{|z|=r} \lambda(z), \text{ for each } 0 \le r < 1.$$

In 1973, Hahn [6, Corollary 1 of Theorem 4] obtained the estimate

$$\beta \ge \frac{K^{1/n}}{4K(2K+1)},$$

for normalized Hahn $K$-mappings.

These classes are related to an important class of holomorphic mappings investigated by Wu [16]. A holomorphic mapping $f$ from the unit ball $B^n$ of $\mathbf{C}^n$ into $\mathbf{C}^n$ is said to be a Wu $K$-mapping if

$$|f'| \le K|\det f'|^{1/n}$$

at each point $z \in B^n$.

We denote by $\mathcal{F}_{K,n}$ the family of all Wu $K$-mappings of $B^n$ into $\mathbf{C}^n$. For $n > 1$ and $K \ge 1$, we define the Bloch constant

$$\beta(K,n) = \inf\{\beta_f : f \in \mathcal{F}_{K,n}, \det f'(0) = 1\},$$

for $n$-dimensional Wu $K$-mappings.

Wu [16] gave a short and elegant proof that the Bloch constant $\beta(K,n)$ is positive for any fixed $K$ and $n$. In fact, as Wu remarks, the existence of a (positive) Bloch constant for Wu $K$-mappings follows already from the result of Bochner which we mentioned earlier [1].

Since a Wu $K$-mapping is a Takahashi $K'$-mapping, for $K' = \sqrt{n}K$, the above result of Sakaguchi yields the following lower estimate:

$$\beta(K,n) \ge \frac{1}{8K^{2n-1}n^{3/2}}\left(1 - \frac{1}{n}\right)^{n-2} > \frac{1}{8eK^{2n-1}n^{3/2}}. \tag{1}$$

Also, since a Wu $K$-mapping is a Hahn $K^n$-mapping, the above result of Hahn yields the following lower estimate:

$$\beta(K,n) \ge \frac{1}{8K^{2n-1}(1 + 1/(2K^n))}. \tag{2}$$

Harris [7] gave the following lower bound:

$$\beta(K,n) \ge \frac{1}{8K^{2n}}. \tag{3}$$

In this paper, we give two new lower bounds for the Bloch constant $\beta(K,n)$ for Wu $K$-mappings. Since holomorphic $K$-quasiregular mappings (to be defined presently) are Wu $K^{1-1/n}$-mappings, we obtain, as corollaries, estimates for the Bloch constant for holomorphic $K$-quasiregular mappings. Our final result will be a direct estimate (not a corollary) of the Bloch constant for holomorphic $K$-quasiregular mappings.

Let us say that a holomorphic mapping $f$ from the unit ball $B^n$ of $\mathbf{C}^n$ into $\mathbf{C}^n$ is $K$-quasiregular if

$$\Lambda \le K\lambda$$

at each point $z \in B^n$. The justification for defining $K$-quasiregular mappings as above is that, for holomorphic mappings, this definition is equivalent to the assertion that the tangent mapping $f'$ maps spheres onto ellipsoids whose major axes are less than $K$ times their minor axes.

Let us define a mapping to be quasiregular if it is $K$-quasiregular for some $K \geq 1$. Similarly, we may define the families of Bochner, Takahashi, Hahn, and Wu mappings. The families of quasiregular, Bochner and Wu mappings coincide. Also, the families of Takahashi and Hahn mappings coincide. Although the definition of a holomorphic $K$-quasiregular mapping looks very much like the definition of a Hahn $K$-mapping, we note that the mapping $f = (f_1, \dots, f_n)$, where $f_j(z) = z_j^2$, is a Hahn $\sqrt{n}$-mapping but, for each $K$, it is not $K$-quasiregular. Thus, the family of Hahn-Takahashi mappings is strictly larger than the family of quasiregular mappings.

It is known [10] that for $n > 1$, a quasiregular holomorphic mapping is locally biholomorphic. It was proved by Titov [15] (see also Marden and Rickman [10]) that entire quasiregular holomorphic mappings are affine. In fact, Poletsky [11] showed that quasiregular holomorphic mappings (in any bounded domain) are rather rigid. On the other hand, there are still quite a few quasiregular holomorphic mappings, since, for example, Poletsky showed that the holomorphic mapping

$$(z_1, z_2) \longmapsto (z_1 + 3^{-1}(z_1 - 1)^{3/2}, z_2)$$

is quasiregular in the unit ball.

Hahn [6, Cor. 1 of Theorem 3] also considered holomorphic mappings of $B^n$ into $\mathbf{C}^n$ satisfying the "boundedness" condition $\Lambda(z) \leq K$ for $z \in B^n$ and $\det f'(0) = 1$, and obtained the lower bound $1/4K^{2n-1}$ on the Bloch constant for this class of mappings. In Section 2, we consider bounded (in the usual sense) mappings and obtain similar results.

We shall make use of the group of all biholomorphic mappings of $B^n$ onto itself, which is denoted by $\mathrm{Aut}(B^n)$. The following results are known [12]:

*(i) For any* $a \in B^n$,

$$\phi_a(z) = \frac{a - P_a z - (1 - |a|^2)^{1/2} Q_a z}{1 - \langle z, a\rangle} \in \mathrm{Aut}(B^n),$$

*where* $\langle z, a\rangle = z_1\bar{a}_1 + \cdots + z_n\bar{a}_n$, $P_a z = \langle z, a\rangle a/\langle a, a\rangle$, $Q_a z = z - P_a z$.

*(ii)* $\phi_a(0) = a$, $\phi_a(a) = 0$, $\phi_a = \phi_a^{-1}$,

$$\phi_a'(0) = -(1 - |a|^2)P_a - (1 - |a|^2)^{1/2} Q_a, \tag{4}$$

$$\phi_a'(a) = -\frac{1}{1 - |a|^2} P_a - \frac{1}{(1 - |a|^2)^{1/2}} Q_a, \tag{5}$$

$$|\det \phi_a'(0)| = (1 - |a|^2)^{(n+1)/2}. \tag{6}$$

*(iii) For any* $\phi \in \mathrm{Aut}(B^n)$ *with* $\phi(a_1) = a_2$,

$$\phi = \phi_{a_2} \circ U \circ \phi_{a_1}, \tag{7}$$

*for some unitary transformation* $U$.

For any $\phi \in \mathrm{Aut}(B^n)$, as a consequence of (4), (5) and (7),

$$|\phi'(z)\theta| \geq \frac{1 - |\phi(z)|^2}{(1 - |z|^2)^{1/2}}, \text{ for } \theta \in \partial B^n, \tag{8}$$

and it follows from (6) and (7) that

$$|\det \phi'(z)| = \left(\frac{1 - |\phi(z)|^2}{1 - |z|^2}\right)^{(n+1)/2}, \text{ for } z \in B^n. \tag{9}$$

## 2. Bounded mappings

In the theory of functions of one variable, we have the following known result:

**Theorem 1** (Landau)**.** *Let $f$ be a holomorphic function in the unit disk $D$, $f(0) = 0$, $f'(0) = \alpha > 0$ and $|f(z)| < M$ for $z \in D$. Then:*

*(i) $f$ is univalent in $D_{r_0}$, where*

$$r_0 = \frac{\alpha}{M + \sqrt{M^2 - \alpha^2}} > \frac{\alpha}{2M};$$

*(ii) for any positive number $r \le r_0$, $f(D_r)$ contains the disk $D_R$, where*

$$R = M\frac{r(\alpha - Mr)}{M - \alpha r} \ge Mrr_0.$$

*Proof.* First assume that $M = 1$. Then, $\alpha = 1$ implies $f(z) = z$, so the conclusions hold obviously. Now, we assume that $\alpha < 1$.

Let $z_0 \in D$ be such that $|z_0| = \rho_0 > 0$ and $f(z_0) = 0$. Applying the Schwarz-Pick lemma to the function $h(z) = f(z)/z$, we have

$$\left|\frac{h(z_0) - h(0)}{1 - h(0)h(z_0)}\right| \le \rho_0.$$

Note that $h(z_0) = 0$ and $h(0) = \alpha$. It follows that $\rho_0 \ge \alpha$. Thus $\rho_0 \ge r_0$ since $r_0 \le \alpha$.

Let $z_1, z_2 \in D$ be such that $0 < |z_1| \le |z_2| = \rho < 1$ and $f(z_1) = f(z_2) = w_0 \neq 0$. Then the function

$$g(z) = \frac{f(z) - w_0}{1 - \overline{w}_0 f(z)} \cdot \frac{1 - \overline{z}_1 z}{z - z_1} \cdot \frac{1 - \overline{z}_2 z}{z - z_2}$$

is holomorphic in $D$ and, by the maximum principle, $|g(z)| \le 1$ for $z \in D$. In particular, we have $|g(0)| = |w_0|/|z_1 z_2| \le 1$ and

$$|w_0| \le |z_1 z_2| \le \rho^2. \tag{10}$$

Applying the Schwarz-Pick lemma to the function $h(z) = f(z)/z$, we see that $h(z_2)$ lies in the closed disk with diameter $[(\alpha - \rho)/(1 - \alpha\rho), (\alpha + \rho)/(1 + \alpha\rho)]$. Thus

$$|w_0| = |f(z_2)| = \rho|h(z_2)| \ge \frac{\rho(\alpha - \rho)}{1 - \alpha\rho}. \tag{11}$$

Combining (10) and (11), we obtain

$$\rho \ge \frac{\alpha}{1 + \sqrt{1 - \alpha^2}}.$$

This proves $(i)$.

To prove $(ii)$, let $r \le r_0$ and consider the function $h(z) = f(z)/z$. We have $h(0) = \alpha$ and $|h(z)| < 1$ for $z \in D$. By the Schwarz-Pick lemma, $h(re^{i\theta})$, for $0 \le \theta \le 2\pi$, lies on the closed disk with diameter $[(\alpha - r)/(1 - \alpha r), (\alpha + r)/(1 + \alpha r)]$, so

$$|h(re^{i\theta})| \ge \frac{\alpha - r}{1 - \alpha r},$$
$$|f(re^{i\theta})| \ge r \cdot \frac{\alpha - r}{1 - \alpha r} = R.$$

We have

$$R \geq r \cdot \frac{\alpha - r_0}{1 - \alpha r_0} = r r_0.$$

This proves $(ii)$ for $M = 1$.

The theorem has been shown for $M = 1$. We can complete the proof by considering the function $f(z)/M$.

Now, we proceed to extend the above theorem to several variables.

**Lemma 1.** *Let $\psi$ be a holomorphic mapping of the unit disk $D$ into $\mathbf{C}^n$ , $|\psi(z)| < M$ for $z \in D$, $\psi(0) = 0$ and $|\psi'(0)| = \alpha > 0$. Then: $(i)$ $\psi$ is univalent in $D_{r_0}$ and $\psi'(z) \neq 0$ for $z \in D_{r_0}$; $(ii)$ for any positive number $r \leq r_0$, $\psi(\partial D_r)$ lies outside of the ball $B^n(0, R)$, where $r_0$ and $R$ are the numbers defined in Theorem 1.*

*Proof.* Let $\psi = (\psi_1, \cdots, \psi_n)$ and

$$h(z) = \frac{1}{\alpha}\left(\overline{\psi_1'(0)}\psi_1(z) + \overline{\psi_2'(0)}\psi_2(z) + \cdots + \overline{\psi_n'(0)}\psi_n(z)\right).$$

Then $h'(0) = \alpha$ and $|h(z)| \leq |\psi(z)| < M$ for $z \in D$ by the Schwarz inequality. Also, $h(0) = 0$. Applying Theorem 1 to the function $h$, we see that $h$ is univalent in $D_{r_0}$, that $h'(z) \neq 0$ for $z \in D_{r_0}$ and that $h(\partial D_r)$ lies outside of $D_R$ for $r \leq r_0$. Now the conclusions follow since the injectivity of $h$ implies that of $\psi$; $h'(z) \neq 0$ implies $\psi'(z) \neq 0$; and $|\psi(z)| \geq |h(z)|$ for $z \in D$. The lemma is proved.

Applying the Cauchy inequality to the function $h$, we have $\alpha = h'(0) \leq M$, that is,

$$|\psi'(0)| \leq M. \tag{12}$$

Note that (12) holds without the assumption $\psi(0) = 0$.

**Lemma 2.** *Let $A$ be an $n \times n$ complex matrix . Then, for any unit vector $\theta \in \partial B^n$, the following inequality holds:*

$$|A\theta| \geq \frac{|\det(A)|}{|A|^{n-1}}. \tag{13}$$

*If $A$ satisfies $|A| \leq K|\det(A)|^{1/n}$, $K \geq 1$, then, for any $\theta \in \partial B^n$, we have*

$$|A\theta| \geq \frac{|\det(A)|^{1/n}}{K^{n-1}}. \tag{14}$$

*Proof.* We may assume that $\det(A) \neq 0$. There exist two unitary matrices $U_1$ and $U_2$ such that $\Lambda = U_1 A U_2$ is a diagonal matrix with positive elements $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then

$$|\det(A)| = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Assume that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. For any $\theta \in \partial B^n$, since $U_1$ and $U_2$ are unitary, we have

$$|A\theta| = |U_1 A U_2 U_2^{-1}\theta| = |\Lambda\omega| \geq \lambda_1 = \frac{|\det(A)|}{\lambda_2 \cdots \lambda_n},$$

where $\omega = U_2^{-1}\theta \in \partial B^n$. Let $\omega$ be the unit vector, whose only nonzero component is the $k$-th. Then, since $U_1$ is unitary,

$$\lambda_k = |\Lambda\omega| = |U_1 A U_2 \omega| = |A U_2 \omega|.$$

Also, since $U_2$ is unitary, $\omega' = U_2\omega$ is also a unit vector since $U_2$ is unitary. Thus,

$$\lambda_k = |A\omega'| \le |A|,$$

and, if $|A| \le K|\det(A)|^{1/n}$, we have

$$\lambda_k \le K|\det(A)|^{1/n}.$$

The lemma follows.

**Lemma 3.** *Let $f$ be a holomorphic mapping of the unit ball $B^n$ into $\mathbf{C}^n$ such that $|f(z)| \le M$ for $z \in B^n$. Then*

$$|f'(z)| \le \frac{M}{1-|z|^2} \tag{15}$$

*and*

$$|\det f'(z)| \le \frac{M^n}{(1-|z|^2)^{(n+1)/2}}. \tag{16}$$

*Proof.* Define $\psi(\zeta) = f(\zeta\theta)$ for a fixed $\theta \in \partial B^n$. Then $|\psi(\zeta)| \le M$ for $\zeta \in D$. By (12), $|f'(0)\theta| = |\psi'(0)| \le M$. Since $\theta$ may be arbitrary, this proves (15) for $z = 0$.

For a fixed $z' \in B^n$, let $\phi$ be a biholomorphic mapping of $B^n$ onto itself which maps 0 into $z'$, and let $F = f \circ \phi$. By (8),

$$|F'(0)| \ge (1-|z'|^2)|f'(z')|.$$

Since (15) has been proved for $z = 0$ and $|F| \le M$, we have

$$|F'(0)| \le M.$$

It follows that

$$|f'(z')| \le \frac{M}{1-|z'|^2}.$$

(15) is proved.

For $z = 0$, (16) is a consequence of (15). For a fixed $z' \in B^n$, let $\phi$ and $F$ be defined above. By (9),

$$|\det \phi'(0)| = (1-|z'|^2)^{(n+1)/2}.$$

Thus

$$|\det F'(0)| = |\det f'(z')||\det \phi'(0)| = (1-|z'|^2)^{(n+1)/2}|\det f'(z')|.$$

Since (16) has been proved for $z = 0$ and $|F(z)| \le M$, we have

$$|\det F'(0)| \le M^n.$$

It follows from the above two inequalities that

$$|\det f'(z')| \le \frac{M^n}{(1-|z'|^2)^{(n+1)/2}}.$$

This proves (16) and the lemma.

**Lemma 4.** *Let $A$ be a holomorphic mapping of $B^n(0,r)$ into the space of $n \times n$ complex matrices. If $A(0) = 0$ and $|A(z)| \le M$ for $z \in B^n(0,r)$, then*

$$|A(z)| \le M|z|/r, \ \textit{for } z \in B^n(0,r).$$

*Proof.* For each $\theta, \theta', \theta'' \in \partial B^n$, let $g(\zeta) = \langle A(\zeta\theta)\theta', \theta''\rangle$. Then, $g$ is a complex-valued function defined on $D_r$, $g(0) = 0$ and $|g(\zeta)| \leq M$. By the classical Schwarz lemma, $|g(\zeta)| \leq M|\zeta|/r$. Since $\theta$, $\theta'$ and $\theta''$ may be arbitrary, we obtain the conclusion of the lemma.

The following lemma is connected to a result obtained by Takahashi [14], [7].

**Lemma 5.** *Let $f$ be a holomorphic mapping of the unit ball $B^n$ into $\mathbf{C}^n$, such that $|\det f'(0)| = \alpha > 0$ and $|f(z)| < M$ for $z \in B^n$. Then $f$ is univalent on $B^n(0, \rho_0)$, where*

$$\rho_0 = \frac{\alpha}{mM^n},$$

*and $m \approx 4.2$ is the minimum of the function $(2 - r^2)/(r(1 - r^2))$, for $0 \leq r \leq 1$.*

*Proof.* By (15),

$$|f'(z) - f'(0)| \leq M\left(1 + \frac{1}{1 - |z|^2}\right) = \frac{M(2 - |z|^2)}{1 - |z|^2}, \text{ for } z \in B^n.$$

The function $(2 - r^2)/(r(1 - r^2))$, for $0 \leq r \leq 1$, attain its minimun $m \approx 4.2$ at $x_0 \approx 0.66$. Then, by the preceding lemma,

$$|f'(z) - f'(0)| \leq mM|z|, \text{ for } |z| \leq x_0.$$

On the other hand, by (13) and (15),

$$|f'(0)\theta| \geq \frac{\alpha}{|f'(0)|^{n-1}} \geq \frac{\alpha}{M^{n-1}}, \text{ for } \theta \in \partial B^n.$$

Note that $x_0 > 1/m$ and, by Lemma 3, $1/m \geq \rho_0$. Now let $z'$and $z''$ be distinct points in $B^n(0, \rho_0)$ with $|z'| \leq |z''| < \rho_0$, and let $[z', z'']$ denote the segment from $z'$ to $z''$. We have

$$|f'(z) - f'(0)| \leq mM|z''| < \frac{\alpha}{M^{n-1}}, \text{ for } z \in [z', z''].$$

Thus, setting

$$dz = \begin{pmatrix} dz_1 \\ dz_2 \\ \vdots \\ dz_n \end{pmatrix},$$

we have

$$|f(z'') - f(z')| \geq \left|\int_{[z',z'']} f'(0)dz\right| - \int_{[z',z'']} |f'(z) - f'(0)||dz| > 0,$$

since

$$\left|\int_{[z',z'']} f'(0)dz\right| \geq \frac{|z' - z''|\alpha}{M^{n-1}},$$
$$\int_{[z',z'']} |f'(z) - f'(0)||dz| < \frac{|z' - z''|\alpha}{M^{n-1}}.$$

The lemma is proved.

**Theorem 2.** *Let $f$ be a holomorphic mapping of the unit ball $B^n$ into $\mathbf{C}^n$ such that $f(0) = 0$, $\det f'(0) = \alpha > 0$, and $|f(z)| < M$ for $z \in B^n$. Let $\rho_0$ be the number defined in the previous lemma and*

$$\rho_1 = \frac{\alpha}{2M^n},$$

$$R_0 = M\rho_0\rho_1 = \frac{\alpha^2}{2mM^{2n-1}}.$$

*Then $f$ maps $B^n(0, \rho_0)$ injectively onto a domain which contains $B^n(0, R_0)$.*

*Proof.* First of all, we have proved that $f$ is univalent on $B^n(0, \rho_0)$.

For any fixed $z' \in \partial B^n(0, \rho_0)$, let $\theta = z'/|z'| = z'/\rho_0 = (\theta_1, \theta_2, \cdots, \theta_n) \in \partial B^n$. Then $z' = \rho_0\theta$. Note that $\rho_0 < 1$, since $\alpha \leq M^n$ by (16). Let $\psi(\zeta) = f(\theta\zeta)$ for $\zeta \in D$. Then, $|\psi(z)| < M$ for $z \in D$, $\psi(0) = 0$, and

$$\psi'(0) = \sum_{k=1}^{n} \theta_k \cdot \frac{\partial f}{\partial z_k}(0) = f'(0)\theta.$$

By (13) and (15),

$$|\psi'(0)| \geq \frac{\alpha}{M^{n-1}}.$$

Let $f = (f_1, f_2, \cdots, f_n)$ and $\psi = (\psi_1, \psi_2, \cdots, \psi_n)$. Let $r_0$ be the number defined in Theorem 1 and Lemma 1 for the function $\psi$. Then

$$r_0 > \rho_1 = \frac{\alpha}{2M^n} > \rho_0.$$

Thus, we can apply Lemma 1 to the function $\psi$ and assert that $|f(z')| = |\psi(\rho_0)| \geq R_0$. This shows that $f(\partial B^n(0, \rho_0))$ lies outside of $B^n(0, R_0)$. The theorem is proved.

By Theorem 1, the function $g(z_1) = Mz_1(z_1 + \beta)/(1 + \beta z_1)$ for $z_1 \in D$, $M > 0$, $0 < \beta < 1$, is univalent in $D_{r_0}$, where

$$r_0 = \frac{\beta}{1 + \sqrt{1 - \beta^2}}.$$

Note that $g'$ has a zero at $z = -r_0$. So the maximum schlicht disk contained in $g(D)$with center $g(0)$ has radius $M\beta^2/\left(1 + \sqrt{1 - \beta^2}\right)^2$. Now, let $f = (f_1, \cdots, f_n)$ be a mapping of $B^n$ into $\mathbf{C}^n$ defined by $f_1(z) = g(z_1)$, $f_j = Mz_j$ for $j = 2, \cdots, n$. Then $f(0) = 0$, $\det f'(0) = M^n\beta$, and $|f(z)| < M$ for $z \in B^n$. The number $R_0$ in the above theorem is equal to $M\beta/(2m)$. On the other hand, the maximum schlicht ball contained in $f(B^n)$ with center $f(0)$ has radius $M\beta^2/\left(1 + \sqrt{1 - \beta^2}\right)^2$. This example shows that the lower bound $R_0$ in Theorem 2 is a reasonable estimate; it is not off by more than a factor of $2m \approx 8.4$.

For Wu $K$-mappings, Theorem 2 has another version.

**Theorem 3.** *Let $f$ be a Wu $K$-mapping of the unit ball $B^n$ into $\mathbf{C}^n$ such that $f(0) = 0$, $|\det f'(0)| = \alpha > 0$, and $|f(z)| < M$ for $z \in B^n$. Then there exists a domain $G \subset B^n(0, \rho_0)$ such that $0 \in G$ and $f$ maps $G$ onto a ball $B^n(0, R_0)$ injectively, where*

$$\rho_0 = \frac{\alpha^{1/n}}{2MK^{n-1}},$$

$$R_0 = M\rho_0^2 = \frac{\alpha^{2/n}}{4MK^{2n-2}}.$$

*Proof.* For $n = 1$, this is due to Landau and follows immediately from Theorem 1. Suppose $n > 1$. For any fixed $\theta \in \partial B^n$, define $\psi(\zeta) = f(\zeta\theta)$. We have

$$\psi'(0) = \sum_{k=1}^{n} \theta_k \cdot \frac{\partial f}{\partial z_k}(0) = f'(0)\theta.$$

Since $f$ is a Wu $K$-mapping, $|f'(0)| \le K|\det f'(0)|^{1/n}$, so by (14) we assert that

$$|\psi'(0)| = |f'(0)\theta| \ge \frac{\alpha^{1/n}}{K^{n-1}}.$$

We indicated that quasiregular holomorphic mappings, for $n > 1$, are locally biholomorphic. So, in particular, $f$ is locally univalent in $B^n(0, \rho_0)$. On the other hand, by Lemma 1, $|f(\zeta\theta)| = |\psi(\zeta)| \ge R_0$ for $\zeta \in \partial D_{\rho_0}$. Now, since $\theta \in \partial B^n$ may be arbitrary, we conclude that $f(\partial B^n(0, \rho_0))$ lies outside of the ball $B^n(0, R_0)$. An application of the monodromy theorem asserts the existence of the domain $G$ which possesses the properties in the theorem. The proof is complete.

## 3. Bloch constants for quasiregular holomorphic mappings

**Theorem 4.** *Let $f$ be a Wu $K$-mapping of the unit ball $B^n$ into $\mathbf{C}^n$, $n > 1$, such that* $\det f'(0) = 1$. *Then*

$$\beta_f \ge \frac{1}{9.83K^{2n-1}} > \frac{1}{10K^{2n-1}}. \tag{17}$$

*Proof.* Without loss of generality, we may assume that $f$ is holomorphic on $\overline{B}^n$. Let $z' \in B^n$ be a point such that $(1-|z'|)^n|\det f'(z')| = 1$ and $(1-|z|)^n|\det f'(z)| \le 1$, for $|z'| = r \le |z| \le 1$. In particular, $|\det f'(z)| \le |\det f'(z')|$ for $|z| = r$. By the maximum principle, $|\det f'(z)| \le |\det f'(z')|$ for $|z| \le r$.

For $\zeta \in B^n$ and fixed $0 < k < 1$, define

$$g(\zeta) = z' + k(1-r)\zeta, \quad F(\zeta) = \frac{1}{k}\left(f(g(\zeta)) - f(z')\right).$$

Then,

$$F(0) = 0, \quad |\det F'(0)| = (1-r)^n|\det f'(z')| = 1.$$

If $|g(\zeta)| \le r$,

$$\begin{aligned}|\det F'(\zeta)| &= (1-r)^n|\det f'(g(\zeta))| \\ &\le (1-r)^n|\det f'(z')| = 1;\end{aligned}$$

and if $|g(\zeta)| \ge r$,

$$\begin{aligned}&|\det F'(\zeta)| = (1-r)^n|\det f'(g(\zeta))| \\ &= \left(\frac{1-r}{1-|g(\zeta)|}\right)^n (1-|g(\zeta)|)^n|\det f'(g(\zeta))| \le \left(\frac{1-r}{1-|g(\zeta)|}\right)^n \\ &\le \left(\frac{1-r}{1-r-k(1-r)|\zeta|}\right)^n = \left(\frac{1}{1-k|\zeta|}\right)^n.\end{aligned}$$

We conclude that

$$|\det F'(\zeta)| \le \left(\frac{1}{1-k|\zeta|}\right)^n, \text{ for } \zeta \in B^n.$$

For $\zeta = s\theta \in B^n$ with $\theta \in \partial B^n$ and positive $s$, we have

$$\begin{aligned}|F(\zeta)| &\le \int_0^{\zeta} |dF(\zeta)| = \int_0^{|\zeta|} |F'(s\theta)\theta| ds \\ &\le K \int_0^{|\zeta|} |\det F(s\theta)|^{1/n} ds \\ &\le K \int_0^{|\zeta|} \frac{ds}{1-ks} < \frac{K}{k} \log \frac{1}{1-k},\end{aligned}$$

since $F$ is also a Wu $K$-mapping. Now, applying Theorem 3 to the function $F$, we see that $F(B^n)$ contains a schlicht ball with center 0 and radius

$$\frac{k}{4K^{2n-1}\log(1/(1-k))}.$$

Consequently, $f(B^n)$ contains the schlicht ball with center $f(z')$ and radius

$$\frac{k^2}{4K^{2n-1}\log(1/(1-k))}.$$

Taking $k = 0.7$, we see that the image $f(B^n)$ contains a schlicht ball of radius $(9.83K^{2n-1})^{-1}$. This proves the theorem.

**Lemma 6.** *Let $\phi$ be a holomorphic mapping of $B^n$ into itself. If $\phi(0) = a$, then*

$$|\phi(z)| \le \frac{|a| + |z|}{1 + |a||z|}, \ for\ z \in B^n.$$

*In particular, for the function $\phi_a(z)$ defined in the introduction, we have*

$$1 - |\phi_a(z)|^2 \ge \frac{(1-|a|^2)(1-|z|^2)}{(1+|a||z|)^2}, \ for\ z \in B^n. \tag{18}$$

The proof of the above lemma is just an application of the classical Schwarz-Pick lemma to the function $\langle \theta_1, \phi(\theta_2\zeta)\rangle$ for $\zeta \in D$, where $\theta_1, \theta_2 \in \partial B^n$ may be arbitrary.

The following lemma was proved by X. Liu and D. Minda [8], [9].

**Lemma 7.** *Let $f$ be a locally univalent holomorphic mapping of $B^n$ into $\mathbf{C}^n$. If* $\det f'(0) = 1$ *and*

$$(1-|z|^2)^{(n+1)/2} |\det f'(z)| \le 1,$$

*then*

$$|\det f'(z)| \ge \frac{1}{(1-|z|)^{n+1}} \exp\left\{-\frac{(n+1)|z|}{1-|z|}\right\}, \ for\ z \in B^n.$$

**Theorem 5.** *Let $f$ be a Wu $K$-mapping of the unit ball $B^n$ into $\mathbf{C}^n$, $n > 1$, such that* $\det f'(0) = 1$. *Then*

$$\beta_f \ge \frac{8^{1/(n+1)} e^{1+1/n}}{4\sqrt{2}K^{n-1}} \int_0^1 \frac{1}{t^{1/2+1/n}} \exp\left\{-\frac{n+1}{n}\frac{1}{t}\right\} dt. \tag{19}$$

*Proof.* As in the proof of Theorem 4, we consider the function

$$F(\zeta) = \frac{1}{k}\left(f(g(\zeta)) - f(z')\right),$$

where $g(\zeta) = z' + k(1-r)\zeta$ and $0 < k < 1$. This time, we take $k = 1/2$. We have proved that

$$|\det F'(0)| = 1, \tag{20}$$

$$|\det F'(\zeta)| < \left(\frac{1}{1-k|\zeta|}\right)^n < 2^n, \text{ for } \zeta \in B^n. \tag{21}$$

Now, we start from the function $F$.

By (20),

$$\max_{\zeta \in B^n} (1 - |\zeta|^2)^{(n+1)/2} |\det F'(\zeta)| = A \geq 1.$$

Let the maximum be attained at a point $a$. Then,

$$|a| \leq \sqrt{1 - 2^{-2n/(n+1)}},$$

since, by (21),

$$(1 - |\zeta|^2)^{(n+1)/2} |\det F'(\zeta)| < 1, \text{ for } |\zeta| \geq \sqrt{1 - 2^{-2n/(n+1)}}.$$

Let $\phi_a \in \operatorname{Aut}(B^n)$ be the biholomorphic mapping of $B^n$ onto itself presented in *(i)* of Section 1, and let $w = G(\omega) = F(\phi_a(\omega))$. We have, by (9),

$$\begin{aligned} |\det G'(0)| &= |\det \phi_a'(0)| |\det F'(a)| = (1 - |a|^2)^{(n+1)/2} |\det F'(a)| = A, \\ (1 - |\omega|^2)^{(n+1)/2} |\det G'(\omega)| &= (1 - |\omega|^2)^{(n+1)/2} |\det \phi_a'(\omega)| |\det F'(\phi_a(\omega))| \\ &= (1 - |\phi_a(\omega)|^2)^{(n+1)/2} |\det F'(\phi_a(\omega))| \leq A, \text{ for } \omega \in B^n. \end{aligned}$$

Recalling that holomorphic quasiregular mappings with $n > 1$ are locally biholomorphic and applying the preceding lemma to the function $G/A^{1/n}$, we have

$$|\det G'(\omega)| \geq \frac{e^{n+1}}{(1-|\omega|)^{n+1}} \exp\left\{-\frac{n+1}{1-|\omega|}\right\}, \text{ for } \omega \in B^n. \tag{22}$$

Now, let $G(0) = w_0$ and let $r_0$ be the supremum of values $r$ such that there is a domain $\Omega_r \subset B^n$ containing 0 which is mapped biholomorphically onto the ball $B^n(w_0, r)$. Then, since $G$ is locally biholomorphic, it follows from the monodromy theorem that there is a point $w_1$ on the boundary of the ball $B^n(w_0, r_0)$ such that the arc $\gamma = G^{-1}[w_0, w_1]$ originates from the origin and tends to $\partial B^n$ as $w \to w_1$. Let $\Gamma = \phi_a(\gamma) = F^{-1}[w_0, w_1]$. We have

$$r_0 = |w_1 - w_0| = \int_\Gamma |dF(\zeta)| = \int_\Gamma \left| F'(\zeta) \frac{d\zeta}{|d\zeta|} \right| |d\zeta|.$$

Since $F$ is also a Wu $K$-mapping, by (14),

$$\left| F'(\zeta) \frac{d\zeta}{|d\zeta|} \right| \geq \frac{|\det F'(\zeta)|^{1/n}}{K^{n-1}}. \tag{23}$$

Thus

$$r_0 \geq \frac{1}{K^{n-1}} \int_\Gamma |\det F'(\zeta)|^{1/n} |d\zeta|.$$

Using (8), (9), (18) and (22), we have

$$\begin{aligned}
&|\det F'(\zeta)|^{1/n}|d\zeta| \\
&\geq |\det F'(\phi_a(\omega))|^{1/n}|\det \phi_a'(\omega)|^{1/n}\frac{(1-|\omega|^2)^{(n+1)/(2n)}}{(1-|\phi_a(\omega)|^2)^{(n+1)/(2n)}}\frac{1-|\phi_a(\omega)|^2}{(1-|\omega|^2)^{1/2}}|d\omega| \\
&= |\det G'(\omega))|^{1/n}|(1-|\phi_a(\omega)|^2)^{(n-1)/(2n)}(1-|\omega|^2)^{1/(2n)}|d\omega| \\
&\geq \frac{e^{(n+1)/n}(1-|a|^2)^{(n-1)/(2n)}(1-|\omega|^2)^{1/2}}{(1-|\omega|)^{(n+1)/n}(1+|a||\omega|)^{(n-1)/n}}\exp\left\{-\frac{n+1}{n}\frac{1}{1-|\omega|}\right\}|d\omega| \\
&= \frac{e^{(n+1)/n}(1-|a|^2)^{(n-1)/(2n)}(1+|\omega|)^{1/2}}{(1-|\omega|)^{(n+2)/(2n)}(1+|a||\omega|)^{(n-1)/n}}\exp\left\{-\frac{n+1}{n}\frac{1}{1-|\omega|}\right\}|d\omega|.
\end{aligned}$$

Since

$$(1-|a|^2)^{(n-1)/(2n)} > \left(\frac{1}{2}\right)^{(n-1)/(n+1)},$$

$$\frac{(1+|\omega|)^{1/2}}{(1+|a||\omega|)^{(n-1)/n}} \geq \left(\frac{1}{(1+|a||\omega|)}\right)^{1/2-1/n} \geq \left(\frac{1}{2}\right)^{1/2-1/n},$$

we have

$$\frac{(1-|a|^2)^{(n-1)/(2n)}(1+|\omega|)^{1/2}}{(1+|a||\omega|)^{(n-1)/n}} \geq \left(\frac{1}{2}\right)^{3/2-3/(n+1)}.$$

Thus

$$r_0 \geq \frac{e^{1+1/n}}{2^{3/2-3/(n+1)}K^{n-1}}\int_0^1 \frac{1}{(1-|\omega|)^{1/2+1/n}}\exp\left\{-\frac{n+1}{n}\frac{1}{1-|\omega|}\right\}d|\omega|.$$

Since $\beta_f \geq r_0/2$, setting $1-|\omega|=t$, the theorem is proved.

Let

$$c_n = \frac{8^{1/(n+1)}e^{1+1/n}}{4\sqrt{2}}\int_0^1 \frac{1}{t^{1/2+1/n}}\exp\left\{-\frac{n+1}{n}\frac{1}{t}\right\}dt.$$

Then, (19) becomes

$$\beta_f \geq c_n K^{-(n-1)}.$$

We have, for $n \geq 2$,

$$c_n > \frac{e}{4\sqrt{2}}\int_0^1 \frac{1}{t^{1/2}}\exp\left\{-\frac{3}{2t}\right\}dt > 0.04.$$

If we set $x = 1/n$, then $c_n = b(x)$, where

$$\begin{aligned}
b(x) &= \frac{8^{x/(x+1)}e^{1+x}}{4\sqrt{2}}\int_0^1 \frac{1}{t^{1/2+x}}\exp\left\{-\frac{1+x}{t}\right\}dt \\
&= \sqrt{2}e\int_0^1 \frac{1}{8^{1/(1+x)}t^{1/2+x}}\exp\left\{x-\frac{1}{t}-\frac{x}{t}\right\}dt \\
&= \sqrt{2}e\int_0^1 t^{-1/2}e^{-1/t}\exp\left\{x-\frac{x}{t}-x\log t-\frac{\log 8}{1+x}\right\}dt.
\end{aligned}$$

For $0 < x \leq 1/2$, we have

$$b'(x) \geq \sqrt{2}e\int_0^1 t^{-1/2}e^{-1/t}\lambda(t)\exp\left\{x-\frac{x}{t}-x\log t-\frac{\log 8}{1+x}\right\}dt,$$

where

$$\lambda(t) = 1 + \frac{4}{3}\log 2 - \frac{1}{t} - \log t.$$

Then $\lambda(t) > 0$ for $t \geq 1/3$, and $\lambda(t) - 0.1 < 0$ for $0 < t \leq 1/3$. Thus, for $0 < x \leq 1/2$,

$$\begin{aligned} b'(x) &\geq \frac{\sqrt{2}e}{8}\int_{1/3}^{1} t^{-1/2}e^{-3/(2t)}\lambda(t)dt + \frac{\sqrt{2}e^{3/2}}{4}\int_0^{1/3} t^{-1}e^{-1/t}(\lambda(t) - 0.1)dt \\ &\geq \frac{\sqrt{2}e}{8}(0.069 - 2\sqrt{e} \times 0.009) > 0. \end{aligned}$$

This shows that $b$ is increasing for $0 < x \leq 1/2$ and $c_n$ is decreasing. Consequently,

$$c_n \geq \lim_{m\to\infty} c_m \geq b(0).$$

A numerical calculation shows that $b(0) \geq 0.0856$. So, we may assert that

$$c_n > 0.0856 > \frac{1}{12}, \text{ for } n \geq 2. \tag{24}$$

From Theorem 4, we have

$$\beta(K, n) \geq \frac{1}{10K^{2n-1}}. \tag{25}$$

It follows from Theorem 5 and (24) that

$$\beta(K, n) \geq \frac{1}{12K^{n-1}}. \tag{26}$$

Now, for the Bloch constant $\beta(K, n)$ for Wu $K$-mappings, we have five lower bounds: (1), (2), (3), (25) and (26). The following example shows that (26) is most reasonable. Let $f(z) = Az$, where $A$ is an $n \times n$ diagonal matrix with positive elements $K, \cdots, K, 1/K^{n-1}$, $K > 1$. Then $f \in \mathcal{F}_{K,n}$ and $\beta_f = 1/K^{n-1}$. However, (26) is not always best. Comparing them with one another, first we see that (25) is always better than (1), and then (2) is worse than (25) if $1 \leq K^n < 2$ and is worse than (26) if $K^n \geq 2$. Finally, no one among (3), (25) and (26), can be covered by the other two.

Let $f$ be a holomorphic $K$-quasiregular mapping. Then,

$$\begin{aligned} |\det f'| &= \lambda_1 \cdots \lambda_n \geq \lambda^{n-1}\Lambda \geq \Lambda^n/K^{n-1}, \\ |f'| &= \Lambda \leq K^{1-1/n}|\det f'|^{1/n}. \end{aligned}$$

Thus, $K$-quasiregular mappings are Wu $K^{1-1/n}$-mappings; and so from the preceding five lower bounds for $\beta_f$, for normalized Wu mappings of the ball, we obtain as corollaries five lower bounds for $\beta_f$, for normalized $K$-quasiregular mappings of the ball.

We conclude this paper with an estimate which applies directly (not as a corollary) to quasiregular mappings.

**Theorem 6.** *Let $f$ be a holomorphic $K$-quasiregular mapping of the unit ball $B^n$ into $\mathbf{C}^n$, $n > 1$, such that* $\det f'(0) = 1$. *Then*

$$\beta_f \geq \frac{8^{1/(n+1)}e^{1+1/n}}{4\sqrt{2}K^{1-1/n}}\int_0^1 \frac{1}{t^{1/2+1/n}}\exp\left\{-\frac{n+1}{n}\frac{1}{t}\right\}dt > \frac{1}{12K^{1-1/n}}.$$

*Proof.* The theorem can be proved by repeating the proof of Theorem 5 word for word, except that the inequality (23) should be replaced by

$$\left| F'(\zeta)\frac{d\zeta}{|d\zeta|} \right| \geq \lambda(\zeta) \geq \frac{|\det F'(\zeta)|^{1/n}}{K^{1-1/n}},$$

since $F$ is also $K$-quasiregular and

$$|\det F'| \leq \lambda\Lambda^{n-1} \leq \lambda^n K^{n-1}.$$

It is easy to show that Wu $K$-mappings are $K^n$-quasiregular, and so Theorem 5 follows directly from Theorem 6.

Since holomorphic $K$-quasiregular mappings are Hahn $K$-mappings, Hahn's estimate

$$\beta_f \geq \frac{K^{1/n}}{4K(2K+1)}$$

holds also for quasiregular holomorphic mappings. Theorem 6 yields a better result for each $K > 1$. If we wish a lower bound which is independent of $n$, Hahn's estimate gives $\beta_f \geq 1/4K(2K+1)$ and our result gives the better estimate $\beta_f \geq 1/12K$. The example $f(z) = (K^{1/n}z_1, \cdots, K^{1/n}z_{n-1}, K^{(1-n)/n}z_n)$ shows that our estimate is best possible in terms of powers of $K$.

Just as we modified the proof of Theorem 5 to obtain Theorem 6, we can modify the proof of Theorem 4 to obtain that for holomorphic quasiregular mappings

$$\beta_f \geq \frac{1}{10K^{3(n-1)/n}}.$$

When $K$ is very close to 1, this is better than Theorem 6.

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